

NOTES ON BANACH FUNCTION SPACES, XV_B

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54. *The ideals $L^n(B)$*

From Theorem 21.3 in Note VI it follows immediately that if L is Dedekind complete, then L^a is equal to the intersection of all $\sigma(L, L^\sim)$ -closed and super dense ideals in L . The purpose of the next three sections is to prove similar results for the ideals $L^a(B)$ and $L^{an}(B)$ in arbitrary Riesz spaces.

Let L be an arbitrary Riesz space and let $A \subset L$ be a normal subspace of L . As usual we shall write $N_A = \{\varphi: \varphi \in L^\sim \text{ and } \varphi(f) = 0 \text{ for all } f \in A\} = A^0$. Then N_A is a normal subspace in L^\sim (Theorem 21.1 (i) in Note VI) and its disjoint complement $C_A = N_A^\perp$ in L^\sim will be called the carrier of A . Since L^\sim is Dedekind complete we have that $L^\sim = N_A \oplus C_A$. The projection of L^\sim onto C_A will be denoted by P_A . It is evident that for every $\varphi \in L^\sim$, $(P_A\varphi)(f) = \varphi(f)$ for all $f \in A$. In particular, if A_u denotes the normal subspace of L which is generated by $0 < u \in L$ and if P_u denotes the projection of L^\sim onto the carrier of A_u in L^\sim , then $(P_u\varphi)(u) = \varphi(u)$ for every $\varphi \in L^\sim$.

Let $\pi = \pi(K_\tau)$ be a system of normal subspaces of L which is directed upwards and which has the additional property that the smallest ideal containing $\cup K_\tau$ is quasi order dense in L . We shall call any system $\pi = \pi(K_\tau)$ of this kind an *exhausting system of normal subspaces of L* . If $\pi = \pi(K_\tau)$ is exhausting, then obviously $\cap K_\tau^\perp = \{0\}$. If π is exhausting and $\pi = \pi(K_n)$ consists of an increasing sequence of normal subspaces of L , then π will be called *sequential*. If every K_τ^\perp of an exhausting system $\pi = \pi(K_\tau)$ has a weak unit, i.e., there is an element $0 < u_\tau \in K_\tau^\perp$ such that the ideal generated by u_τ is quasi order dense in K_τ^\perp , then π is called a *principal exhausting system* or shortly *principal*.

Let $\pi = \pi(K_\tau)$ be an exhausting system of normal subspaces of L . Then by $\pi(P_\tau)$ we shall denote the corresponding system of projections of L^\sim onto the carriers of the disjoint compliments K_τ^\perp . Observe that $\pi(P_\tau)$ is directed downwards and that the projection $P^\pi = \inf P_\tau$ (i.e., $P^\pi\varphi = \inf P_\tau\varphi$ for all $0 < \varphi \in L^\sim$) is the projection of L^\sim onto the normal subspace $C^\pi = \cap_\tau C_\tau$, where C_τ denotes the carrier of K_τ^\perp .

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After the above preliminaries we shall introduce now the following definition.

Definition 54.1. Let $\pi = \pi(K_\tau)$ be an exhausting system of normal subspaces of L and let $B \subset L^\sim$ be an ideal. Then by $L^\pi(B)$ we shall denote the set of all $f \in L$ such that $\inf (P_\tau \varphi)(|f|) = 0$ for all $0 \leq \varphi \in B$. If $B = L^\sim$, then we shall write L^π in place of $L^\pi(L^\sim)$.

It is obvious that if $B = \{0\}$, then $L^\pi(B) = L$ for all exhausting systems π . We shall now prove the following important theorem.

Theorem 54.2. For every exhausting system $\pi = \pi(K_\tau)$ and every ideal $B \subset L^\sim$, the set $L^\pi(B)$ has the following properties. (i) $L^\pi(B) = {}^0(B \cap C^\pi)$, and so $L^\pi(B)$ is an ideal. (ii) $K_\tau \subset L^\pi(B)$ for all τ , and so $L^\pi(B)$ is quasi order dense. (iii) $C^\pi \subset L_0^\sim$, and so $L^\pi(B)$ is $\sigma(L, B \cap L_0^\sim)$ -closed.

Proof. (i) Observe that if $0 < \varphi \in C^\pi$, then $P_\tau \varphi = \varphi$ for all τ . Indeed, this follows from $C^\pi = \bigcap C_\tau$. Thus if $f \in L^\pi(B)$ and $0 < \varphi \in B \cap C^\pi$, then $\varphi(|f|) = (P^\pi \varphi)(|f|) = \inf (P_\tau \varphi)(|f|) = 0$, i.e., $f \in {}^0(B \cap C^\pi)$. Conversely, assume that $f \in {}^0(B \cap C^\pi)$ and assume that $0 < \varphi \in B$. Then $\inf_\tau (P_\tau \varphi)(|f|) = (P^\pi \varphi)(|f|) = 0$ since $P^\pi \varphi \in B \cap C^\pi$, and so $f \in L^\pi(B)$. (ii) If $0 < u \in K_\tau$ and $K_\sigma \supset K_\tau$, then, by Theorem 27.6 in Note VIII applied to L^\sim , $K_\tau \perp K_\sigma^\perp$ implies that $(P_\sigma \varphi)(u) = P_\tau(P_\sigma \varphi)(u) = 0$ for all $0 < \varphi \in B$, and so $K_\tau \subset L^\pi(B)$ for all τ . (iii) From (i) and (ii) it follows that if $\varphi \in C^\pi$, then $L^\pi \subset N_\varphi$ and hence N_φ is quasi order dense, or equivalently $\varphi \in L_0^\sim$. Thus $C^\pi \subset L_0^\sim$. Since (i) implies that $L^\pi(B)$ is $\sigma(L, B \cap C^\pi)$ -closed and $C^\pi \subset L_0^\sim$ we obtain that $L^\pi(B)$ is also $\sigma(L, B \cap L_0^\sim)$ -closed.

In all the statements of the following theorem π is an arbitrary exhausting system and B is an arbitrary ideal in L^\sim .

Theorem 54.3. (i) $L^\pi(B)$ is $\sigma(L, L^\sim)$ -closed.

(ii) $L^\pi(B \cap L_c^\sim)$ is σ -normal, i.e., $0 \leq u_n \in L^\pi(B \cap L_c^\sim)$ and $u_n \uparrow u$ implies $u \in L^\pi(B \cap L_c^\sim)$.

(iii) $L^\pi(B) = L^\pi(B \cap L_0^\sim)$.

(iv) If L is Archimedean or if the weaker hypothesis $L_0^\sim \subset L_{sn}^\sim$ holds, then $C^\pi \subset L_{sn}^\sim$, and so $L^\pi(B) = L^\pi(B \cap L_{sn}^\sim) = L^\pi(B \cap L_{sn}^\sim \cap L_0^\sim)$, or equivalently, $L^\pi(B \cap L_n^\sim) = L$.

Proof. (i) follows immediately from Theorem 54.2 (iii).

(ii) Observe that if $\varphi \in L_c^\sim$, then N_φ is σ -normal and apply Theorem 54.2 (i).

(iii) From $C^\pi \subset L_0^\sim$ it follows immediately that $L^\pi(B) = {}^0(B \cap C^\pi) = {}^0(B \cap L_0^\sim \cap C^\pi) = L^\pi(B \cap L_0^\sim)$.

(iv) $C^\pi \subset L_0^\sim \subset L_{sn}^\sim$ implies that $B \cap C^\pi = B \cap L_{sn}^\sim \cap C^\pi = B \cap L_{sn}^\sim \cap L_0^\sim \cap C^\pi$, and $B \cap L_n^\sim \cap C^\pi = \{0\}$. Thus (iv) follows immediately from Theorem 54.2 (i).

A few more elementary properties of $L^\pi(B)$ are listed in the following theorem. In all statements π is an arbitrary exhausting system and B with or without subscript denotes an arbitrary ideal in L^\sim .

Theorem 54.4. (i) $B \subset B_1$ implies that $L^\pi(B_1) \subset L^\pi(B)$. In particular, $L^\pi \subset L^\pi(B)$ for all $B \subset L^\sim$.

(ii) $L^\pi(B \oplus B_1) = L^\pi(B) \cap L^\pi(B_1)$ and $L^\pi(B \cap B_1) \supset L^\pi(B) \oplus L^\pi(B_1)$.

(iii) $L^\pi(B) = L^\pi(\{B\})$, where $\{B\}$ is the normal subspace generated by B .

Proof. (i) is trivial. (ii) It is obvious from (i) that $L^\pi(B \oplus B_1) \subset L^\pi(B) \cap L^\pi(B_1)$. Let $0 < u \in L^\pi(B) \cap L^\pi(B_1)$ and let $0 < \varphi \in B \oplus B_1$. Then $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 \in B$ and $\varphi_2 \in B_1$. Hence, $(P_\pi \varphi)(u) = (P_\pi \varphi_1)(u) + (P_\pi \varphi_2)(u)$ implies that $u \in L^\pi(B \oplus B_1)$. The second part of (ii) follows immediately from (i). In order to prove (iii) observe that $B \cap C^\pi$ is order dense in $\{B\} \cap C^\pi$ and apply Lemma 52.4 (ii) and Theorem 54.2 (i).

We shall now give another characterization of the ideals $L^a(B)$, $L^{an}(B)$ and $L^{acn}(B)$ along the same lines as the characterization of L^a in Theorem 21.3 in Note VI.

Theorem 54.5. Let L be an arbitrary Riesz space and let $B \subset L$ be an arbitrary ideal in L^\sim .

(i) $L^a(B) = \cap (L^\pi(B \cap L_s^\sim): \pi \text{ is exhausting}) = \cap (L^\pi(B \cap L_s): \pi \text{ is exhausting and sequential}) = \cap (L^\pi(B \cap L_s^\sim): \pi \text{ is exhausting, sequential and principal})$.

(ii) $L^{an}(B) = \cap (L^\pi(B \cap L_{sn}^\sim): \pi \text{ is exhausting}) = \cap (L^\pi(B \cap L_{sn}^\sim): \pi \text{ is exhausting and principal})$.

(iii) $L^{acn}(B) = \cap (L^\pi(B \cap L_{sn,c}^\sim): \pi \text{ is exhausting}) = (L^\pi(B \cap L_{sn,c}^\sim): \pi \text{ is exhausting and principal})$.

Proof. (i) We shall first show that $L^a(B) \subset L^\pi(B \cap L_s^\sim) = L^\pi(B \cap L_s^\sim \cap L_0^\sim)$. From Theorem 54.2 (i) it follows that $L^\pi(B \cap L_s^\sim) = {}^0(B \cap L_s^\sim \cap C^\pi)$. Since ${}^0(B \cap L_s^\sim) \subset {}^0(B \cap L_s^\sim \cap C^\pi)$ and $L^a(B) = {}^0(B \cap L_s^\sim)$ by Theorem 52.3 (i) we obtain that $L^a(B) \subset L^\pi(B \cap L_s^\sim)$. In order to complete the proof of (i) we shall prove that $A = \cap (L^\pi(B \cap L_s^\sim): \pi \text{ is exhausting, sequential and principal}) \subset L^a(B)$. To this end, let $0 \leq u_n \uparrow u \in A$ and let $0 \leq \varphi \in B$. Then we have to show that $\varphi(u - u_n) \downarrow 0$. Let $0 < \alpha < 1$, let A_n be the normal subspace of L generated by $(u_n - \alpha u)^-$ and let $K_n = A_n^p (n = 1, 2, \dots)$. Then $\pi = \pi(K_n)$ is sequential and principal. In order to show that π is exhausting observe that $A_u \oplus A_{u^p}$, where A_u is the ideal generated by u , is quasi order dense in L by Lemma 48.2 (v). Since $(A_u)^p \subset K_n$ for all n and $K_n \ni (u_n - \alpha u)^+ \uparrow (1 - \alpha)u$ we see that the ideal generated by $\cup K_n$ is order dense in $A_u \oplus A_{u^p}$, and so is quasi order dense in L . Hence, $u \in L^\pi(B \cap L_s^\sim)$, and so $(P_n \varphi)(u) \downarrow 0$ for all $0 \leq \varphi \in B \cap L_s^\sim$. Now observe that $(u_n - \alpha u)^- \in K_n^p$ and $(u_n - \alpha u)^- \leq u$ imply that for every $0 < \varphi \in L^\sim$, $\varphi((u_n - \alpha u)^-) = (P_n \varphi)((u_n - \alpha u)^-) \leq (P_n \varphi)(u)$.

Then $u - u_n = (1 - \alpha)u + \alpha u - u_n = (1 - \alpha)u + (u_n - \alpha u)^- - (u_n - \alpha u)^+ \leq (1 - \alpha)u + (u_n - \alpha u)^-$ implies that $\varphi(u - u_n) \leq (1 - \alpha)\varphi(u) + (P_n\varphi)(u)$ for all n and all $0 < \varphi \in B$. From $u \in L^\pi(B)$ it follows then that $\varphi(u - u_n) \downarrow 0$, and so $u \in L^a(B)$. This completes the proof of (i).

Since the proofs of (ii) and (iii) are word for word the same we will not give them here.

Corollary 54.6. (i) $L^a(B)$ is the intersection of all the $\sigma(L, B \cap L_s^-)$ -closed and quasi order dense ideals.

(ii) $L^{an}(B)$ is equal to the intersection of all the $\sigma(L, B \cap L_{sn}^-)$ -closed and quasi order dense ideals.

(iii) $L^{acn}(B)$ is equal to the intersection of all the $\sigma(L, B \cap L_{sn,c}^-)$ -closed and quasi order dense ideals.

Proof. We shall only indicate how to prove (i) since the proofs of (ii) and (iii) are similar. Since $L^\pi(B \cap L_s^-)$ is quasi order dense and $\sigma(L, B \cap L_s^-)$ -closed, we have only to show that if A is a $\sigma(L, B \cap L_s^-)$ -closed ideal and quasi order dense, then $L^a(B) \subset A$. But this is an easy consequence of the fact that if A is $\sigma(L, B \cap L_s^-)$ -closed, then $A = \cap(N_\varphi: \varphi \in B \cap L_s^- \text{ and } A \subset N_\varphi)$ and the fact that $L^a(B) = {}^0(B \cap L_s^-)$ by Theorem 52.3 (ii).

Theorem 54.7. If L is Archimedean or if the weaker hypothesis $L_0^- \subset L_{sn}^-$ holds, then $L^{an}(B) = \cap(L^\pi(B): \pi \text{ is exhausting}) = \cap(L^\pi(B): \pi \text{ is exhausting and principal})$ and $L^{acn}(B) = \cap(L^\pi(B \cap L_c^-): \pi \text{ is exhausting}) = \cap(L^\pi(B \cap L_c^-): \pi \text{ is exhausting and principal})$.

Proof. From Theorem 54.3 (iv) it follows that $L^\pi(B) = L^\pi(B \cap L_{sn}^-)$ provided that $L_0^- \subset L_{sn}^-$, and so the present characterization follows from Theorem 54.5 (ii). Observe that, by Theorem 54.4 (ii), $L^\pi(B \cap L_c^-) = L^\pi(B \cap L_{sn,c}^-) \cap L^\pi(B \cap L_n^-)$. Since by Theorem 54.3 (iv), $L_0^- \subset L_{sn}^-$ implies that $L^\pi(B \cap L_n^-) = L$ we obtain that in that case $L^\pi(B \cap L_c^-) = L^\pi(B \cap L_{sn,c}^-)$, and so the present characterization of $L^{acn}(B)$ follows from Theorem 54.5 (iii).

Whether $L^\pi(B) = L^\pi(B \cap L_s^-)$ holds for every exhausting system which is also sequential without additional hypothesis on L other than $L_0^- \subset L_{sn}^-$ is not known to the present author. The following theorem contains a result which is in that direction.

Theorem 54.8. If the Riesz space L has the property that there exists a projection of L on every normal subspace which is generated by a single element, then $L^\pi(B) = L^\pi(B \cap L_s^-)$ for every ideal $B \subset L^-$ and every exhausting system which is sequential and principal, and so in that case $L^a(B) = \cap(L^\pi(B): \pi \text{ is exhausting, sequential and principal})$.

Proof. Let $\pi = \pi(K_n)$ be exhausting and principal. By Theorem 54.4 (i) we have that $L^\pi(B) \subset L^\pi(B \cap L_s^-)$. In order to prove that these two

ideals are equal we have only to show that $L^\pi(B) \subset N_\varphi$ and $\varphi \in L_e^\sim$ implies that $\varphi = 0$. Indeed, in that case, $L^\pi(B) = {}^0(B \cap L_s^\sim \cap C^\pi) = L^\pi(B \cap L_s^\sim)$. We will show this by proving that $L^\pi(B)$ is super order dense (i.e., for every $0 < u \in L$ there exists a sequence $0 \leq u_n \in L^\pi(B)$ ($n = 1, 2, \dots$) such that $u_n \uparrow u$). To this end, observe that since π is principal there exists by hypothesis a projection on every K_n . Hence, $\cup K_n$ is super order dense, and so, by Theorem 54.2 (i), $L^\pi(B)$ is also super order dense in L . This completes the proof of the theorem.

From the proof of Theorem 54.8 we obtain immediately the following corollary.

Corollary 54.9. *If L is Dedekind complete, then $L^\pi(B) = L^\pi(B \cap L_s^\sim)$ for every ideal $B \subset L^\sim$ and every π which is exhausting and sequential, and so in that case $L^a(B) = \cap (L^\pi(B) : \pi \text{ is exhausting and sequential}) = \cap (L^\pi(B) : \pi \text{ is exhausting, sequential and principal})$.*

We shall conclude this section with a proof that Theorem 21.3 in Note VI is contained in Corollary 54.9.

If L is Dedekind complete and if $A \subset L$ is a normal subspace of L , then for every $f \in L$ and every $\varphi \in L^\sim$ we have $(P_A \varphi)(f) = \varphi(f_A)$, where P_A is the projection of L^\sim onto the carrier C_A of A in L^\sim and where f_A is the component of f in A . Indeed, if $L = A \oplus A^\perp$, then $L^\sim = C_A \oplus C_{A^\perp}$ follows from the observation that $C_A \oplus C_{A^\perp}$ is a normal subspace in L^\sim (Theorem 17.2 in Note VI) and if $L^\sim \ni \varphi \perp C_A \oplus C_{A^\perp}$, then $\varphi \in N_A \cap N_{A^\perp}$, and so for all $f \in L$, $\varphi(f) = \varphi(f_A) + \varphi(f_{A^\perp}) = 0$ which implies that $C_A \oplus C_{A^\perp} = L^\sim$.

Let $\pi = \pi(K_n)$ be a system of normal subspaces of L which is exhausting and sequential. If L is Dedekind complete, then $f = f_{\pi n} + f'_{\pi n}$ with $f_{\pi n} \in K_n$ and $f'_{\pi n} \in K_n^\perp$ for all $f \in L$. In section 21 of Note VI we considered the set L^π of all $f \in L$ such that $\varphi(f'_{\pi n}) \rightarrow 0$ as $n \rightarrow \infty$ for every π which is exhausting and sequential and every $\varphi \in L^\sim$. Since from the remark made above it follows that $\varphi(f'_{\pi n}) = P_n(\varphi)(f)$ we obtain that $L^\pi = \cap (L^\pi : \pi \text{ is exhausting and sequential})$. Thus Theorem 21.3 in Note VI which states that $L^\pi = L^a$ also follows from Corollary 54.9 with $B = L^\sim$.

55. The ideals L_ϱ^π

Let L be a Riesz space and let ϱ be a Riesz seminorm defined on L . If $\pi = \pi(K_\tau)$ is an exhausting system of normal subspaces of L , then we set $L_\varrho^\pi = L^\pi(L_\varrho^*)$, $L_{\varrho,s}^\pi = L^\pi(L_{\varrho,s}^*)$, $L_{\varrho,sn}^\pi = L^\pi(L_{\varrho,sn}^*)$, $L_{\varrho,c}^\pi = L^\pi(L_{\varrho,c}^*)$ and $L_{\varrho,sn,c}^\pi = L^\pi(L_{\varrho,sn,c}^*)$.

The following theorem is an immediate consequence of Theorem 54.5.

Theorem 55.1. *Let L be a Riesz space and let ϱ be a Riesz seminorm on L .*

(i) $L_\varrho^\pi = \cap (L_{\varrho,s}^\pi : \pi \text{ is exhausting}) = \cap (L_{\varrho,s}^\pi : \pi \text{ is exhausting and sequential}) = \cap (L_{\varrho,s}^\pi : \pi \text{ is exhausting, sequential and principal})$.

(ii) $L_\varrho^{an} = \cap (L_{\varrho, sn}^\pi: \pi \text{ is exhausting}) = \cap (L_{\varrho, sn}^\pi: \pi \text{ is exhausting and principal})$.

(iii) $L_\varrho^{acn} = \cap (L_{\varrho, sn, c}^\pi: \pi \text{ is exhausting}) = \cap (L_{\varrho, sn, c}^\pi: \pi \text{ is exhausting and principal})$.

If ϱ is a Riesz norm, then L_ϱ is Archimedean and so Theorem 54.7 gives the following result.

Theorem 55.2. *If L is a Riesz space and if ϱ is a Riesz norm on L , then $L_\varrho^{an} = \cap (L_\varrho^\pi: \pi \text{ is exhausting}) = \cap (L_\varrho^\pi: \pi \text{ is exhausting and principal})$, and $L_\varrho^{acn} = \cap (L_{\varrho, c}^\pi: \pi \text{ is exhausting}) = \cap (L_{\varrho, c}^\pi: \pi \text{ is exhausting and principal})$.*

From Theorem 53.7 we obtain the following result.

Theorem 55.3. *If L_ϱ is a normed Riesz space such that ϱ satisfies (A, iii) or L_ϱ is σ -Dedekind complete, then $L_\varrho^a = L_\varrho^{an} = \cap (L_\varrho^\pi: \pi \text{ is exhausting, sequential and principal})$.*

We shall now turn our attention to the condition that $L_\varrho^{an} = L_\varrho^\pi$ for some π .

Theorem 55.4. *Let L_ϱ be a normed Riesz space. If $L_\varrho^{an} = L_\varrho^\pi$ for some exhausting system π , then the conditions of Theorem 53.9 are fulfilled.*

Proof. $L_\varrho^{an} = L_\varrho^\pi$ implies that L_ϱ^{an} is order dense in L_ϱ , and so the required result follows from Theorem 53.10 (ii).

Let $L_\varrho = l_\infty$. Then $L_\varrho^a = L_\varrho^{an} = c_0 = L_\varrho^\pi$, $\pi = \pi(K_n)$, where K_n is the normal subspace generated by the characteristic function of the set $\{1, 2, \dots, n\}$ ($n = 1, 2, \dots$). Of more interest is the fact that in Example 53.12 we also have that $L_\varrho^a = L_\varrho^{an} = c_0 \cap \{f: f(0) = 0\} = L_\varrho^\pi$, $\pi = \pi(K_\tau)$, where K_τ is the normal subspace generated by the characteristic function of a finite subset of X not containing zero. Indeed, the only thing we have to verify is that $u \in L_\varrho^\pi$ implies that $u(0) = 0$. Let u_τ be the component of u in K_τ^\perp . Then for every $\varphi \in L_\varrho^*$ we have that $(P_\tau \varphi)(u) = \varphi(u_\tau)$. Let $\varphi(f) = f(0)$, $f \in L$. Then $\varphi(u_\tau) = u_\tau(0) = u(0)$ for all τ and so $u(0) \neq 0$ implies that $u \notin L_\varrho^\pi$.

Theorem 55.5. *If L_ϱ is σ -Dedekind complete and if $L_\varrho^a = L_\varrho^{an} = L_\varrho^\pi$ for some exhausting π which is sequential, then L_ϱ is super Dedekind complete, and so $L_{\varrho, c}^* = L_{\varrho, n}^*$.*

Proof. Since L_ϱ is σ -Dedekind complete, it follows from Theorem 53.7 that $L_\varrho^a = L_\varrho^{an}$ is a super Dedekind complete ideal in L_ϱ . From $L_\varrho^a = L_\varrho^{an} = L_\varrho^\pi$, π is sequential and exhausting and L_ϱ is σ -Dedekind complete we conclude as in the proof of Theorem 54.8 that $L_\varrho^a = L_\varrho^{an}$ is super order dense in L_ϱ . Then finally L_ϱ is super Dedekind complete follows from Theorem 29.13 in Note IX.

Since the normed Riesz space L_ϱ in Example 53.12 is σ -Dedekind complete but not Dedekind complete we have that $L_\varrho^a = L_\varrho^{an} \neq L_\varrho^\pi$ for

every π which is exhausting and sequential. From Theorem 54.8 it follows, however, that $L_e^a = L_e^{an} = \bigcap (L_e^\pi : \pi \text{ is exhausting and sequential})$.

A combination of Theorem 25.6 in Note VII and Theorem 55.5 gives the following result.

Theorem 55.6. *Let L_e be σ -Dedekind complete and let L_e have a countable order basis and let $L_e^a = L_e^{an}$ be order dense in L_e . Then L_e is super Dedekind complete if and only if $L_e^a = L_e^{an} = L_e^\pi$ for some π which is exhausting and sequential.*

56. A generalization of Andô's theorem

In Note XIV we have shown among other things that Andô's theorem (Theorem 47.3 in Note XIV) implies that if L is an arbitrary Riesz space and if B is an ideal in L^\sim with the property that N_φ is normal for all $\varphi \in B$, then $B \subset L_n^\sim$ (Theorem 47.5 in Note XIV). In other words, using the notation introduced in section 53, $L^{an}(B) = L$ if and only if N_φ is normal for every $\varphi \in B$. If the Riesz space L is Archimedean, then this result follows immediately from Theorem 54.7 which states that $L^{an}(B) = \bigcap (L^\pi(B) : \pi \text{ is exhausting})$. Indeed, if N_φ is normal for every $\varphi \in B$, then $L^\pi(B) = \bigcap (N_\varphi : \varphi \in B \cap C^\pi)$ is normal and since L is Archimedean it follows from Theorem 54.2 (i) and Theorem 29.10 in Note IX that $L^\pi(B)$ is order dense in L , and so $L^\pi(B) = L$ for every π .

The purpose of this section is to show that Theorem 54.5 can be strengthened in such a way that among other things it will imply Andô's theorem.

Let L be an arbitrary Riesz space and let $0 < u \in L$. Then for every directed system $0 \leq u_\tau \uparrow u$ and every $0 < \alpha < 1$ we shall denote by $\pi = \pi(\{u_\tau\}, \alpha)$ the exhausting system $\pi = \pi(K_\tau)$, where K_τ is equal to the disjoint complement of the ideal generated by the element $(u_\tau - \alpha u)^-$. That such a system is exhausting was already shown in the proof of Theorem 54.5. If the directed system is an increasing sequence $0 \leq u_n \uparrow u$, then we shall write $\pi = \pi(\{u_n\}, \alpha)$, $0 < \alpha < 1$, for the corresponding exhausting system $\pi = \pi(K_n)$, where K_n is the disjoint complement of the ideal generated by the element $(u_n - \alpha u)^-$.

In what follows A_u will always denote the ideal generated by the element $0 < u \in L$. Furthermore, we recall that an ideal $A \subset L$ is called super order dense in L if for every $0 < u \in L$ there exists a sequence $u_n \in A$ ($n = 1, 2, \dots$) such that $0 \leq u_n \uparrow u$.

Lemma 56.1. (i) *For every $0 < u \in L$ and every ideal $B \subset L^\sim$, $L^\pi(B) \cap A_u$ is super order dense in A_u for all $\pi = \pi(\{u_n\}, \alpha)$, $0 < \alpha < 1$.*

(ii) *For every $0 < u \in L$ and every ideal $B \subset L^\sim$, $L^\pi(B) \cap A_u$ is order dense in A_u for all $\pi = \pi(\{u_\tau\}, \alpha)$, $0 < \alpha < 1$.*

Proof. Observe that $(u_n - \alpha u)^+ \in L^\pi(B) \cap A_u$ ($n = 1, 2, \dots$) for all $\pi = \pi(\{u_n\}, \alpha)$, $0 < \alpha < 1$, and that $(u_n - \alpha u)^+ \uparrow (1 - \alpha)u$. Thus $L^\pi(B) \cap A_u$

is super order dense in A_u . Similarly, $(u_\tau - \alpha u)^+ \in L^\pi(B) \cap A_u$ for all $\pi = \pi(\{u_\tau\}, \alpha)$, $0 < \alpha < 1$, and $(u_\tau - \alpha u)^+ \uparrow (1 - \alpha)u$ shows that $L^\pi(B) \cap A_u$ is order dense in A_u .

The following theorem is the main theorem of this section.

Theorem 56.2. *Let L be an arbitrary Riesz space. Then for every $0 < u \in L$ we have*

- (i) $L^a(B) \cap A_u = \cap(L^\pi(B \cap \widetilde{L}_s): \pi = \pi(\{u_n\}, \alpha), 0 < \alpha < 1) \cap A_u$;
- (ii) $L^{an}(B) \cap A_u = \cap(L^\pi(B \cap \widetilde{L}_{sn}): \pi = \pi(\{u_\tau\}, \alpha), 0 < \alpha < 1) \cap A_u$.

Proof. Since always $L^a(B) \subset L^\pi(B \cap \widetilde{L}_s)$ we have only to show that if $0 < v \in \cap(L^\pi(B \cap \widetilde{L}_s): \pi = \pi(\{u_n\}, \alpha), 0 < \alpha < 1)$ and $0 < v \leq u$, then $v \in L^a(B)$. To this end, we have to show that if $0 < v_n \uparrow v$ and $0 < \varphi \in B \cap \widetilde{L}_s$, then $\varphi(v - v_n) \downarrow 0$. Let $u - v + v_n = w_n (n = 1, 2, \dots)$. Then $0 \leq w_n \uparrow u$. Consider now the exhausting system $\pi = \pi(\{w_n\}, \alpha)$, $0 < \alpha < 1$. Then as in the proof of Theorem 54.5 we have that $v - v_n = u - w_n \leq (1 - \alpha)u + (w_n - \alpha u)^- (n = 1, 2, \dots)$, and so if $0 < \varphi \in B \cap \widetilde{L}_s$, then $\varphi(v - v_n) \leq (1 - \alpha)\varphi(u) + \varphi((w_n - \alpha u)^-) = (1 - \alpha)\varphi(u) + (P_n\varphi)((w_n - \alpha u)^-)$. Since $(w_n - \alpha u)^- = (v - v_n - (1 - \alpha)u)^+ \leq v$ we obtain that $\varphi(v - v_n) \leq (1 - \alpha)\varphi(u) + (P_n\varphi)(v) (n = 1, 2, \dots)$. Then the assumption $v \in \cap(L^\pi(B \cap \widetilde{L}_s): \pi = \pi(\{u_n\}, \alpha), 0 < \alpha < 1)$ implies that $(P_n\varphi)(v) \downarrow 0$, and so $\varphi(v - v_n) \downarrow 0$. Thus $v \in L^a(B)$ and the proof of (i) is finished. The proof of (ii) is word for word the same.

It is evident that

$$\begin{aligned} \cap(\cap(L^\pi(B \cap \widetilde{L}_s): \pi = \pi(\{u_n\}, \alpha), 0 < \alpha < 1): 0 < u \in L) &= L^a(B) \text{ and} \\ \cap(\cap(L^\pi(B \cap \widetilde{L}_{sn}): \pi = \pi(\{u_\tau\}, \alpha), 0 < \alpha < 1): 0 < u \in L) &= L^{an}(B). \end{aligned}$$

Corollary 56.3. *Let L be an arbitrary Riesz space. Then for every $0 < u \in L$ we have*

- (i) $L^a(B) \cap A_u = \cap(N_\varphi: \varphi \in B \cap \widetilde{L}_s \text{ and } N_\varphi \cap A_u \text{ is super order dense in } A_u) \cap A_u$.
- (ii) $L^{an}(B) \cap A_u = \cap(N_\varphi: \varphi \in B \cap \widetilde{L}_{sn} \text{ and } N_\varphi \cap A_u \text{ is order dense in } A_u) \cap A_u$.

Proof. We shall only prove (i) since the proof of (ii) is similar. From Theorem 54.2 (i) it follows that $L^\pi(B \cap \widetilde{L}_s) = \cap(N_\varphi: \varphi \in B \cap \widetilde{L}_s \cap C^\pi)$, and so if $\pi = \pi(\{u_n\}, \alpha)$, $0 < \alpha < 1$, then by Lemma 56.1 (i) every element $\varphi \in B \cap \widetilde{L}_s \cap C^\pi$ has the property that $N_\varphi \cap A_u$ is super order dense in A_u . Then the desired result follows from Theorem 52.3.

From Corollary 56.3 we can deduce the following interesting generalization of Theorem 51.2.

Theorem 56.4. *Let L be an arbitrary Riesz space.*

- (i) *For every $0 < u \in L$, the set of all $\varphi \in L_s$ such that $N_\varphi \cap A_u$ is super order dense in A_u is an order dense ideal in \widetilde{L}_s .*

(ii) For every $0 < u \in L$, the set of all $\varphi \in L_{sn}^{\sim}$ such that $N_{\varphi} \cap A_u$ is order dense in A_u is an order dense ideal in L_{sn}^{\sim} .

Proof. (i) It is easy to see that the set of all $\varphi \in L_s^{\sim}$ such that $N_{\varphi} \cap A_u$ is super order dense in A_u is an ideal. If $0 < \varphi_0 \in L_s^{\sim}$, then by Theorem 52.3 (i) we have that $L^a(B_{\varphi_0}) = N_{\varphi_0}$, and so by Corollary 56.3 (i) we obtain that $N_{\varphi_0} \cap A_u = \cap (N_{\varphi} : \varphi \in B_{\varphi_0})$ and $N_{\varphi} \cap A_u$ is super order dense in A_u , where B_{φ_0} is the ideal generated by φ_0 . Since $\varphi \in B_{\varphi_0}$ implies that $|\varphi| \leq a\varphi_0$ for some real $a > 0$ we obtain immediately that $N_{\varphi_0} \cap A_u = \cap (N_{\varphi} : 0 \leq \varphi \leq \varphi_0)$ and $N_{\varphi} \cap A_u$ is super order dense in A_u . If $\varphi_0(u) = 0$, then $N_{\varphi_0} \cap A_u = A_u$ implies that $N_{\varphi_0} \cap A_u$ is super order dense in A_u . If $\varphi_0(u) \neq 0$, then $N_{\varphi_0} \cap A_u = \cap (N_{\varphi} : 0 \leq \varphi \leq \varphi_0)$ and $N_{\varphi} \cap A_u$ is super order dense in A_u implies that there is an element $0 < \varphi \leq \varphi_0$, $\varphi(u) > 0$ and $N_{\varphi} \cap A_u$ is super order dense in A_u . Since L^{\sim} is Dedekind complete, and hence Archimedean, it follows that the ideal of all $\varphi \in L_s^{\sim}$ such that $N_{\varphi} \cap A_u$ is super order dense in A_u is itself order dense in L^{\sim} . The proof of (ii) is similar.

We shall now show that the following theorem, which contains Andô's theorem, is an immediate consequence of Corollary 56.3.

Theorem 56.5. Let B be an ideal in L .

(i) If for every $\varphi \in B$ the null ideal N_{φ} is σ -normal, i.e., $0 \leq u_n \in N_{\varphi}$ and $u_n \uparrow u$ implies $u \in N_{\varphi}$, then $L^a(B) = L$, or equivalently, $B \subset L_c^{\sim}$.

(ii) If for every $\varphi \in B$ the null ideal N_{φ} is normal, then $L^{an}(B) = L$, or equivalently, $B \subset L_n^{\sim}$.

Proof. (i) If N_{φ} is σ -normal and $N_{\varphi} \cap A_u$ is super order dense in A_u , then $u \in N_{\varphi}$, and so by Corollary 56.3 (i) we have that $u \in L^a(B)$ for all $0 < u \in L$.

(ii) If N_{φ} is normal and $N_{\varphi} \cap A_u$ is order dense in A_u , then $u \in N_{\varphi}$, and so by Corollary 56.3 (ii) we have that $u \in L^{an}(B)$ for all $0 < u \in L$.

For normed spaces we have the following result.

Theorem 56.6. Let L be a Riesz space and let ϱ be a Riesz seminorm on L .

(i) If N_{φ} is σ -normal for every $\varphi \in L_c^*$, then ϱ satisfies (A, i), i.e., $u_n \downarrow 0$ implies $\varrho(u_n) \downarrow 0$.

(ii) If N_{φ} is normal for every $\varphi \in L_c^*$, then ϱ satisfies (A, ii), i.e., $u_n \downarrow 0$ implies $\varrho(u_n) \downarrow 0$.

Proof. (i) follows immediately from Theorem 56.5 (i) and Lemma 22.6 in Note VII.

(ii) follows immediately from Theorem 56.4 (ii) and Corollary 53.3.

Theorem 56.6 (ii) is Andô's theorem (Theorem 47.3 (ii) in Note XIV). The proof, however, is different.

57. *The normal component of an order bounded linear functional*

In Theorem 20.4 of Note VI it was shown that if $0 \leq u \in L$ and $0 \leq \varphi \in L^\sim$ are given, the integral component φ_c of φ satisfies

$$(1) \quad \varphi_c(u) = \inf (\lim \varphi(u_n): 0 \leq u_n \uparrow u).$$

In the present section a similar formula will be proved for the normal component φ_n of φ .

Given $0 \leq u \in L$ and $0 \leq \varphi \in L^\sim$, we set

$$(2) \quad \bar{\varphi}(u) = \inf (\sup \varphi(u_\tau): 0 < u_\tau \uparrow u).$$

Theorem 57.1. (i) *Given $0 \leq \varphi \in L^\sim$, we have $\bar{\varphi}(u) \geq 0$ and $\bar{\varphi}(u+v) = \bar{\varphi}(u) + \bar{\varphi}(v)$ for all $0 \leq u, v \in L$, and hence $\bar{\varphi}$ can be uniquely extended to a positive linear functional on L .*

(ii) *Given $0 \leq \varphi \in L^\sim$, we have $0 \leq \varphi_n \leq \bar{\varphi} \leq \varphi_c \leq \varphi$, where φ_n and φ_c are the normal and integral component of φ respectively.*

Proof. (i) It follows immediately from the definition of $\bar{\varphi}$ that $\bar{\varphi}(u) \geq 0$ for all $0 \leq u \in L$. If $0 \leq u_\tau \uparrow u$ and $0 \leq v_\sigma \uparrow v$, then $0 \leq u_\tau + v_\sigma \uparrow u + v$, and so $\bar{\varphi}(u+v) \leq \bar{\varphi}(u) + \bar{\varphi}(v)$. Now, let $0 \leq w_\tau \uparrow u+v$ such that $\sup \varphi(w_\tau) < \bar{\varphi}(u+v) + \varepsilon$. We set $u_\tau = \inf(u, w_\tau)$ and $v_\tau = w_\tau - u_\tau$. Then $0 \leq u_\tau \uparrow u$ and $0 \leq v_\tau = \inf(u+v, w_\tau) - \inf(u, w_\tau) \leq u+v-u = v$. Furthermore, $w_\tau \leq w_\sigma$ implies that $v_\tau = w_\tau - \inf(u, w_\tau) \leq w_\sigma - \inf(u, w_\sigma) = v_\sigma$, so $0 \leq v_\tau \uparrow v$. In order to prove that v is the least upper bound of $\{v_\tau\}$, assume that $v_\tau \leq z$ for all τ . Then $w_\tau - u_\tau \leq z$, i.e., $w_\tau \leq u_\tau + z \leq u + z$ for all τ , and hence $u+v \leq u+z$, so $v \leq z$. We have, therefore, that $0 \leq v_\tau \uparrow v$. It follows now from $\varphi(u_\tau) + \varphi(v_\tau) = \varphi(w_\tau) \leq \bar{\varphi}(u+v) + \varepsilon$ that $\bar{\varphi}(u) + \bar{\varphi}(v) \leq \bar{\varphi}(u+v) + \varepsilon$. The final result is that $\bar{\varphi}(u+v) = \bar{\varphi}(u) + \bar{\varphi}(v)$.

(ii) Observe first that if $\varphi \geq 0$ is a normal integral, then $\bar{\varphi} = \varphi$ by the definition of $\bar{\varphi}$. Secondly if $0 \leq \varphi_1 \leq \varphi_2$, then $0 \leq \bar{\varphi}_1 \leq \bar{\varphi}_2$. Hence, it follows from $0 \leq \varphi_n \leq \varphi$ that $0 \leq \varphi_n \leq \bar{\varphi}$. Comparing the formulas (1) and (2) above, we obtain $\bar{\varphi} \leq \varphi_c \leq \varphi$.

It follows from this theorem that the mapping $T: \varphi \rightarrow \bar{\varphi}$ is a mapping of $(L^\sim)^+$ into $(L^\sim)^+$, leaving each positive normal integral invariant. We list some further properties of T .

Theorem 57.2. (i) *If $0 \leq \varphi_1 \leq \varphi_2$, then $0 \leq T\varphi_1 \leq T\varphi_2$.*

(ii) *If $\varphi_1, \varphi_2 \geq 0$, then $T(\varphi_1 + \varphi_2) = T\varphi_1 + T\varphi_2$. If $\varphi \geq 0$ and $\alpha \geq 0$, then $T(\alpha\varphi) = \alpha T\varphi$.*

(iii) *If $0 \leq \varphi_1 \leq \varphi_2$, then $T(\varphi_2 - \varphi_1) = T\varphi_2 - T\varphi_1$.*

(iv) *If $\varphi_1, \varphi_2 \geq 0$ and $\varphi_1 \perp \varphi_2$, then $T\varphi_1 \perp T\varphi_2$.*

Proof. (i) is evident from the definition of T .

(ii) For any $0 \leq u \in L$ we have

$$\begin{aligned} T(\varphi_1 + \varphi_2)(u) &= \inf \{ \sup (\varphi_1 + \varphi_2)(u_\tau): 0 \leq u_\tau \uparrow u \} = \\ &= \inf (\sup \{ \varphi_1(u_\tau) + \varphi_2(u_\tau) \}: 0 \leq u_\tau \uparrow u) \geq (T\varphi_1)(u) + (T\varphi_2)(u). \end{aligned}$$

For the converse inequality, let $0 \leq u_\tau \uparrow u$ and let $0 \leq v_\sigma \uparrow u$ such that $\sup \varphi_1(u_\tau) < (T\varphi_1)(u) + \varepsilon$ and $\sup \varphi_2(v_\sigma) < (T\varphi_2)(u) + \varepsilon$. Then, since $w_{\tau,\sigma} = \inf(u_\tau, v_\sigma) \uparrow u$ and $\sup_{\tau,\sigma} \varphi_i(w_{\tau,\sigma}) < (T\varphi_i)(u) + \varepsilon$ for $i=1, 2$, we have

$$T(\varphi_1 + \varphi_2)(u) \leq \sup (\varphi_1 + \varphi_2)(w_{\tau,\sigma}) < (T\varphi_1)(u) + (T\varphi_2)(u) + 2\varepsilon.$$

The equality $T(a\varphi) = aT\varphi$ is evident.

(iii) Follows immediately from (ii).

(iv) Since $T\varphi_1 \leq \varphi_1$ and $T\varphi_2 \leq \varphi_2$ it follows from $\varphi_1 \perp \varphi_2$ that $T\varphi_1 \perp T\varphi_2$.

Now, let $\varphi \in L^\sim$; $\varphi = \varphi^+ - \varphi^-$. Since $\varphi^+ \perp \varphi^-$, we have $T\varphi^+ \perp T\varphi^-$, and so $T\varphi^+ - T\varphi^-$ is the decomposition into positive and negative parts of an element of L^\sim , which by definition we shall denote by $T\varphi$. Hence $T\varphi = T\varphi^+ - T\varphi^-$. Observe that for $\varphi \geq 0$ this is in accordance with the earlier notations. It follows that $(T\varphi)^+ = T(\varphi^+)$ and $(T\varphi)^- = T(\varphi^-)$, and so $|T\varphi| = T|\varphi|$.

Theorem 57.3. (i) We have $T(\varphi_1 + \varphi_2) = T\varphi_1 + T\varphi_2$ for $\varphi_1, \varphi_2 \in L$. If $\varphi \in L^\sim$ and a is a real number, then $T(a\varphi) = aT\varphi$. It follows that T is a linear mapping of L^\sim into L^\sim , leaving the elements of L_n^\sim invariant.

(ii) If $\varphi_\tau \uparrow \varphi$ in L^\sim , then $T\varphi_\tau \uparrow T\varphi$.

(iii) If $\varphi_1, \varphi_2 \in L^\sim$, then $T\{\sup(\varphi_1, \varphi_2)\} = \sup(T\varphi_1, T\varphi_2)$ and $T\{\inf(\varphi_1, \varphi_2)\} = \inf(T\varphi_1, T\varphi_2)$.

Proof. (i) Let first $\varphi = \psi_1 - \psi_2$ with $\psi_1, \psi_2 \geq 0$. Observing that $\varphi^+ = (\psi_1 - \psi_2)^+ = \psi_1 - \inf(\psi_1, \psi_2)$ and $\varphi^- = (\psi_1 - \psi_2)^- = \psi_2 - \inf(\psi_1, \psi_2)$ and using part (iii) of the preceding theorem, we obtain

$$T\varphi = T\{\psi_1 - \inf(\psi_1, \psi_2)\} - T\{\psi_2 - \inf(\psi_1, \psi_2)\} = T\psi_1 - T\psi_2.$$

Now, let $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1, \varphi_2 \in L^\sim$. Then

$$\varphi = (\varphi_1^+ + \varphi_2^+) - (\varphi_1^- + \varphi_2^-) = \psi_1 - \psi_2$$

with $\psi_1, \psi_2 \geq 0$, so

$$T\varphi = T\psi_1 - T\psi_2 = (T\varphi_1^+ + T\varphi_2^+) - (T\varphi_1^- + T\varphi_2^-) = T\varphi_1 + T\varphi_2.$$

The equality $T(a\varphi) = aT\varphi$ is evident.

(ii) Given that $\varphi_\tau \uparrow \varphi$ in L^\sim , we have $\varphi - \varphi_\tau \geq T(\varphi - \varphi_\tau) = T\varphi - T\varphi_\tau \geq 0$, and so $T\varphi_\tau \uparrow T\varphi$.

(iii) By the linearity of T the desired result follows immediately from $T\varphi^+ = (T\varphi)^+$ and $T\varphi^- = (T\varphi)^-$.

In summary, T is a linear mapping of L^\sim into L^\sim which preserves the lattice operations and which leaves L_n^\sim fixed. We shall now prove that T is a projection, namely the projection of L^\sim onto L_n^\sim .

Theorem 57.4. *Let L be an arbitrary Riesz space. Then T is the projection of L^\sim onto L_n^\sim . In other words, for every $0 \leq \varphi \in L^\sim$ and every $0 \leq u \in L$,*

$$\varphi_n(u) = \inf (\sup \varphi(u_\tau): 0 \leq u_\tau \uparrow u).$$

Proof. From $\varphi = \varphi_n + \varphi_{sn}$ and $T\varphi = T\varphi_n + T\varphi_{sn} = \varphi_n + T\varphi_{sn}$ it follows that we have to show that for every $0 < \varphi \in L_{sn}^\sim$, $T\varphi = \bar{\varphi} = 0$. If $0 < \varphi \in L_{sn}^\sim$, then by Theorem 56.4 (ii) we have for every $0 < u \in L$ that

$$\varphi = \sup (\psi: 0 \leq \psi \leq \varphi \text{ and } N_\psi \cap A_u \text{ is order dense in } A_u).$$

Hence, by Theorem 57.3 (ii) we have for every $0 < u \in L$ that

$$\bar{\varphi} = \sup (\bar{\psi}: 0 \leq \bar{\psi} \leq \varphi \text{ and } N_{\bar{\psi}} \cap A_u \text{ is order dense in } A_u).$$

From $N_\psi \cap A_u$ is order dense in A_u it follows that there is a directed system $0 \leq u_\tau \in N_\psi$ for all τ and $u_\tau \uparrow u$, and so $\bar{\varphi}(u) = 0$. Hence $\bar{\varphi}(u) = 0$ for every $0 < u \in L$. This completes the proof of the theorem.

Corollary 57.5. *Let L be an arbitrary Riesz space and let $0 \leq \varphi \in L^\sim$. Then the elements $0 \leq u \in L$ satisfying $\varphi_n(u) = \varphi(u)$ are exactly those with the property that for every directed system $0 \leq u_\tau \uparrow u$ the equality $\sup \varphi(u_\tau) = \varphi(u)$ holds.*

In the following theorem we shall give a summary of the various formulas for the integral component, the singular component, the normal component and the singular normal component of a positive linear functional respectively.

Theorem 57.6. *Let L be an arbitrary Riesz space. Then for every $0 \leq \varphi \in L^\sim$ and for every $0 < u \in L$ the following statements hold.*

- (i) $\varphi_c(u) = \inf (\lim \varphi(u_n): 0 \leq u_n \uparrow u) = \inf (\lim \varphi(u_n): 0 \leq u_n \uparrow v \geq u)$
 $= \inf (\sum \varphi(u_n): u \leq \sum u_n) = \inf (\sum \varphi(u_n): u = \sum u_n) =$
 $= \inf (\lim \varphi(u_n): 0 \leq u_n \uparrow \leq u \text{ and } |\psi|(u - u_n) \downarrow 0 \text{ for all } \psi \in L_c^\sim).$
- (ii) $\varphi_s(u) = \sup (\lim \varphi(u_n): u \geq u_n \downarrow 0).$
- (iii) $\varphi_n(u) = \inf (\sup \varphi(u_\tau): 0 \leq u_\tau \uparrow u) = \inf (\sup \varphi(u_\tau): 0 \leq u_\tau \uparrow v \geq u) =$
 $= \inf (\sup \varphi(u_\tau): 0 \leq u_\tau \uparrow \leq u \text{ and } |\psi|(u - u_\tau) \downarrow 0 \text{ for all } \psi \in L_n^\sim).$
- (iv) $\varphi_{sn}(u) = \sup (\inf \varphi(u_\tau): u \geq u_\tau \downarrow 0).$

Proof. (i) We have only to show that $\varphi_0(u) = \inf (\lim \varphi(u_n): 0 \leq u_n \uparrow \leq u \text{ and } |\psi|(u - u_n) \downarrow 0 \text{ for all } \psi \in L_c^\sim)$ is equal to $\varphi_c(u)$. It is obvious that $\varphi_0(u) \leq \varphi_c(u)$. Furthermore, $\varphi_{c,0}(u) = \varphi_c(u)$, and so $\varphi_c \leq \varphi$ implies that $\varphi_c(u) \leq \varphi_0(u)$, which proves that $\varphi_0(u) = \varphi_c(u)$.

(ii) $\varphi_s(u) = \varphi(u) - \varphi_c(u) = \varphi(u) - \inf (\lim \varphi(u_n): 0 \leq u_n \uparrow u) = \varphi(u) + \sup (\lim -\varphi(u_n): 0 \leq u_n \uparrow u) = \sup (\lim \varphi(-u_n): 0 \leq u_n \uparrow u) = \sup (\lim \varphi(u_n): u \geq u_n \downarrow 0).$

The proofs of (ii) and (iii) are similar.

In Theorem 20.6 in Note VI we have shown that if L has the Egoroff property, then in the formula $\varphi_c(u) = \inf (\lim \varphi(u_n): 0 \leq u_n \uparrow u), 0 < u \in L$, for the integral component of a positive linear functional φ the greatest lower bound is attained.

Theorem 57.7. *Let L be an arbitrary Riesz space and let $0 \leq \varphi \in L^\sim$.*

(i) *In order that in the formula $\varphi_c(u) = \inf (\lim \varphi(u_n): 0 \leq u_n \uparrow u)$ the greatest lower bound is attained for all $0 \leq u \in L$ it is necessary and sufficient that N_{φ_s} is super order dense in L .*

(ii) *In order that in the formula $\varphi_n(u) = \inf (\sup \varphi(u_\tau): 0 \leq u_\tau \uparrow u)$ the greatest lower bound is attained for all $0 \leq u \in L$ it is necessary and sufficient that $N_{\varphi_{sn}}$ is order dense in L .*

Proof. The easy proof is left to the reader.

From Theorem 50.4 and the preceding theorem we obtain immediately the following theorem.

Theorem 57.8. *If ${}^0(L_c^\sim) = \{0\}$, then for every positive linear functional the greatest lower bound in the formula for the normal component is everywhere attained.*

The following theorem gives some information about the sets on which the greatest lower bounds are attained. The simple proof is due to J. Holbrook.

Theorem 57.9. *Let L be an arbitrary Riesz space and let $0 < \varphi \in L^\sim$.*

(i) *The set of all $f \in L$ for which there exists a sequence $0 \leq u_n \uparrow (n = 1, 2, \dots)$ such that $u_n \uparrow |f|$ and $\varphi_c(|f|) = \lim \varphi(u_n)$ is an ideal.*

(ii) *The set of all $f \in L$ for which there exists a directed system $0 \leq u_\tau \uparrow \leq |f|$, such that $u_\tau \uparrow |f|$ and $\varphi_n(|f|) = \sup \varphi(u_\tau)$ is an ideal.*

Proof. (i) It is obvious that the only thing we have to show is that if $0 < u \in L$ and $0 \leq u_n \uparrow u$ such that $\varphi_c(u) = \lim \varphi(u_n)$, then the same property holds for every $0 < v < u$. To this end, let $v_n = \inf (u_n, v)$. Then $\varphi(v_n) = \varphi(u_n) - \varphi(\sup (u_n, v) - v)$ for all $n = 1, 2, \dots$, and so $\varphi_c(v) \leq \lim \varphi(v_n) \leq \lim \varphi(u_n) - \varphi_c(u - v) = \varphi_c(v)$, i.e., $\lim \varphi(v_n) = \varphi_c(v)$.

The proof of (ii) is similar.

From Theorem 56.4 we deduce now the following result.

Theorem 57.10. *Let L be an arbitrary Riesz space.*

(i) *For every $0 < u \in L$, the set of all $\varphi \in L^\sim$ such that in the formula for $|\varphi|_c(u)$ the greatest lower bound is attained is an order dense ideal in L^\sim .*

(ii) *For every $0 < u \in L$, the set of all $\varphi \in L^\sim$ such that in the formula for $|\varphi|_n(u)$ the greatest lower bound is attained is an order dense ideal in L^\sim .*

Finally from Corollary 51.2 we obtain the following theorem.

Theorem 57.11. *Let L be an Archimedean Riesz space. Then the set of all $\varphi \in L^\sim$ for which the greatest lower bound is attained in the formula for $|\varphi|_n(u)$ for all $0 < u \in L$ is an order dense ideal in L^\sim .*

Example 57.12. Let $L = C(X)$ be the super Dedekind Riesz space which was introduced in Example 43.5 in Note XIV. Let $\varphi(f) = \sum_1^\infty f(x_n)/2^n$, where $\{x_n: n=1, 2, \dots\}$ is a countable dense subset of X . Since φ is strictly positive and $\varphi_c = \varphi_n = 0$ it follows that in the formula for $\varphi_c = \varphi_n$ the greatest lower bound is nowhere attained in L^+ .

58. Representations of projections in L^\sim

Let L be an arbitrary Riesz space and let B be a normal subspace in L^\sim . If $B = L_c^\sim$ or $B = L_n^\sim$, then in Theorem 57.6 we showed that the B -component of every element $0 < \varphi \in L^\sim$ can be given in terms of a simple formula. This leads to the question whether there are other normal subspaces $B \subset L^\sim$ for which such formulas hold. In view of Theorem 57.6 (i) it could be conjectured that for every normal subspace $B \subset L^\sim$, and every element $0 < \varphi \in L^\sim$,

$$(*) \quad (P_B \varphi)(u) = \inf (\lim \varphi(u_n): 0 \leq u_n \uparrow \leq u \text{ and } \varphi(u - u_n) \downarrow 0 \text{ for all } 0 < \varphi \in B)$$

holds for all $0 < u \in L$, where P_B denotes the projection of L^\sim onto B . Although this formula holds for $B = L_c^\sim$ (Theorem 57.6 ii)) it is obviously false for $B = L_n^\sim$ in a Riesz space L which has the properties ${}^0(L_n^\sim) = \{0\}$ and $L_c^\sim \neq L_n^\sim$. (The Riesz space of Example 53.12 satisfies these conditions). Indeed, in that case $0 \leq u_n \uparrow \leq u$ and $\varphi(u - u_n) \downarrow 0$ for all $\varphi \in B$ implies that $u_n \uparrow u$, and so if $0 < \varphi \in L_{\hat{m},c}^\sim$, then $P_B \varphi = 0$ but the formula gives φ . In (*) we could then try to replace the sequences by arbitrary directed systems. But that does not give us the required result in all cases either. By way of example, let $L = C(0, 1)$ and let B be the normal subspace in L^\sim of all the discrete bounded linear functionals, i.e., $\varphi \in B$ if and only if there exists a sequence $0 \leq x_n \leq 1 (n=1, 2, \dots)$ and a sequence of real numbers $a_n (n=1, 2, \dots)$ such that $\sum |a_n| < \infty$ and $\varphi(f) = \sum a_n f(x_n)$, $f \in L$. In this case, however, if $0 \leq u_\tau \uparrow \leq u$ and $\varphi(u - u_\tau) \downarrow 0$ for all $0 < \varphi \in B$, then $u_\tau(x) \uparrow u(x)$ for all $0 \leq x \leq 1$, and so by Dini's theorem $u_\tau(x) \uparrow u(x)$ on $0 \leq x \leq 1$ uniformly. Hence, for every $0 < \varphi \in L^\sim$ and every $0 < u \in L$, we have

$$\varphi(u) = \inf (\sup \varphi(u_\tau): 0 \leq u_\tau \uparrow \leq u \text{ and } \varphi(u - u_\tau) \downarrow 0 \text{ for all } 0 \leq \varphi \in B).$$

If $\varphi(f) = \int_0^1 f dx$, then $P_B \varphi = 0$, so that again the formula does not give the required result.

Despite the above counter examples we will show in this section that apart from L_c^\sim and L_n^\sim there are other normal subspaces in L^\sim for which (*) holds.

In order to prove (*) or for that matter any formula of this kind we need a criterion which is necessary and sufficient in order that an element $\varphi \in L^\sim$ belongs to the normal subspace B . For a special class of normal subspaces we shall be able to give a criterion of this type. But first we shall introduce some necessary preliminaries.

Let L be an arbitrary Riesz and let M be an arbitrary non-empty subset of L^\sim . We shall say that an element $\varphi \in L^\sim$ is M -absolutely continuous whenever $0 \leq u_k \downarrow$ and $|\psi|(u_k) \downarrow 0$ for all $\psi \in M$ implies that $\inf_k |\varphi(u_k)| = 0$. It is evident that if $0 \leq u_k \downarrow$ and $|\psi|(u_k) \downarrow 0$ for all $\psi \in M$, then also $|\psi|(u_k) \downarrow 0$ for all ψ which are contained in the ideal generated by M . Furthermore, if $0 \leq \psi_\tau \uparrow \psi$, $0 \leq u_k \downarrow$ and $\psi_\tau(u_k) \downarrow 0$ for all τ , then $\psi(u_k) \downarrow 0$, and so we obtain that if φ is M -absolutely continuous, then φ is also B_M -absolutely continuous, where B_M denotes the smallest normal subspace in L^\sim which contains M .

We shall now first prove the following lemma which is a parallel of Theorem 20.1 in Note VI.

Lemma 58.1. *Let M be a non-empty subset of L^\sim and let $\varphi \in L^\sim$. Then the following conditions are mutually equivalent.*

- (i) φ is M -absolutely continuous.
- (ii) φ^+ and φ^- are M -absolutely continuous.
- (iii) $|\varphi|$ is M -absolutely continuous.

Proof. (iii) \Leftrightarrow (ii) and (ii) \Rightarrow (i) are trivial. In order to prove (i) \Rightarrow (ii), let $0 \leq u_k \downarrow$, $|\psi|(u_k) \downarrow 0$ for all $\psi \in M$ and let $0 \leq v \leq u_1$. Then $v - \inf(v, u_k) \leq u_1 - u_k$ and the definition of φ^+ imply that $\varphi(v) - \varphi(\inf(v, u_k)) \leq \varphi^+(u_1) - \varphi^+(u_k)$, and so $0 \leq \varphi^+(u_k) \leq \varphi^+(u_1) - \varphi^+(v) + |\varphi(\inf(v, u_k))|$. Since also $|\psi|(\inf(v, u_k)) \downarrow 0$ for all $\psi \in M$ we obtain that $0 \leq \limsup \varphi^+(u_k) \leq \varphi^+(u_1) - \varphi^+(v)$ for all $0 \leq v \leq u_1$, and hence $\varphi^+(u_k) \downarrow 0$. Thus φ^+ is M -absolutely continuous.

Let $B \subset L^\sim$ be a normal subspace. Then we shall say that B has *locally a countable order basis* if for every $0 < u \in L$, there exists a countable subset of positive elements $\varphi_n (n=1, 2, \dots)$ in B such that for every $0 < \psi \in B$, $\psi(v) = \sup_n (\inf(\psi, n\varphi_n)(v))$ for all $v \in A_u$, where A_u is the ideal generated by u .

We shall now prove the following interesting result.

Theorem 58.2. (Radon-Nikodym Theorem). *If B is a normal subspace of L^\sim which has locally a countable order basis, then $\varphi \in L^\sim$ and φ is B -absolutely continuous implies that $\varphi \in B$.*

Proof. If $\varphi \in L^\sim$ and $\varphi \notin B$, then $|\varphi| \notin B$, and so $\varphi_0 = |\varphi| - P_B|\varphi|$ satisfies $\varphi_0 \perp B$ and φ_0 is B -absolutely continuous. The latter follows from Lemma 58.1. In order to prove the theorem we have to show that $\varphi_0 = 0$. To this end, let $0 < u \in L$ and let A_u be the ideal generated by u . Then, since B has locally a countable order basis, there exists a countable

family of positive elements $\psi_n (n=1, 2, \dots)$ in B which is order dense in B on A_u . Then for every $f \in A_u$ we set $\varphi_0(f) = \sum_{n=1}^{\infty} \psi_n(f)/2^n(1+\psi_n(u))$. From $\varphi_0 \perp B$ it follows that $\inf(\varphi_0, \psi_0)(u) = 0$. Hence, there exist sequences $\{u_k\}, \{u'_k\}$ such that $u = u_k + u'_k$ and $\varphi_0(u_k) + \psi_0(u'_k) \leq 2^{-k}$ for all $k=1, 2, \dots$. Then $\sum \psi_0(u'_k) < \infty$ implies, by Lemma 43.2 in Note XIV, that for every $0 < \varepsilon$ there are sequences $\{v_k\}, \{w_k\}, \{z_k\}$ ($k=1, 2, \dots$) such that $u'_k = v_k + w_k$ for all k , $\varphi_0(w_k) \leq \varepsilon$ for all k and $v_k \leq z_k \downarrow \leq u$ and $\psi_0(z_k) \downarrow 0$. Now $\psi_0(z_k) \downarrow 0$ implies that $\psi_n(z_k) \downarrow 0$ for all $n=1, 2, \dots$, and so, by what we remarked at the beginning, $\varphi_0(z_k) \downarrow 0$ since φ_0 is B -absolutely continuous. Hence, $2^{-k} \geq \varphi_0(u_k) = \varphi_0(u) - \varphi_0(v_k) - \varphi_0(w_k)$ ($k=1, 2, \dots$) implies $0 \leq \varphi_0(u) \leq 2^{-k} + \varphi_0(v_k) + \varepsilon \leq 2^{-k} + \psi_0(z_k) + \varepsilon$ for all $k=1, 2, \dots$. Thus $\varphi_0(u) = 0$ and the proof is finished.

Let B be a normal subspace of L^\sim . Given $0 < u \in L$ and $0 < \varphi \in L^\sim$, we set

$$\bar{\varphi}(u) = \inf(\lim \varphi(u_n): 0 \leq u_n \uparrow \leq u \text{ and } \varphi(u - u_n) \downarrow 0 \text{ for all } 0 \leq \psi \in B).$$

Theorem 58.3. *Let $0 < \varphi \in L^\sim$. Then $\bar{\varphi}$ can be extended uniquely to a positive linear functional on L . Furthermore, $0 \leq \bar{\varphi} \leq \varphi$ and $\bar{\varphi}$ is B -absolutely continuous.*

Proof. It is obvious that $0 \leq \bar{\varphi} \leq \varphi$. For the proof that $\bar{\varphi}$ is additive on $(L^\sim)^+$, and so, by Lemma 18.1 in Note VI, can be extended uniquely to a positive linear functional $\bar{\varphi}$ on all of L we refer the reader to the proof of Theorem 20.4 in Note VI.

In order to prove that $\bar{\varphi}$ is B -absolutely continuous we observe that $\bar{\varphi}(u) = \inf(\sum_1^\infty \varphi(u_n): 0 \leq u_n, u_1 + \dots + u_n \leq u \text{ } (n=1, 2, \dots) \text{ and } \varphi(u) = \sum_1^\infty \varphi(u_n) \text{ for all } 0 < \psi \in B)$. Furthermore, we observe that from the linearity of $\bar{\varphi}$ it follows that $\bar{\varphi}$ is B -absolutely continuous if and only if

$$0 < u_n, \quad u_1 + \dots + u_n \leq u \quad (n=1, 2, \dots)$$

and $\varphi(u) = \sum_1^\infty \varphi(u_n)$ for all $0 < \psi \in B$ implies $\varphi(u) \leq \sum_1^\infty \varphi(u_n)$. To this end, let $\varepsilon > 0$ be given. Then for every n ($n=1, 2, \dots$) there exists

$$0 \leq u_{n,k}, \quad u_{n,1} + \dots + u_{n,k} \leq u_n \quad (k=1, 2, \dots)$$

such that $\varphi(u_n) = \sum_{k=1}^\infty \varphi(u_{n,k})$ for all $0 < \psi \in B$ and

$$\sum_{k=1}^\infty \varphi(u_{n,k}) < \bar{\varphi}(u_n) + \varepsilon/2^n \quad (n=1, 2, \dots).$$

Then $w_n = \sum_{j=1}^n \sum_{k=1}^n u_{j,k} \leq u$ for all $n=1, 2, \dots$ and $\sum_{j,k=1}^\infty \varphi(u_{j,k}) = \varphi(u)$ for all

$0 \leq \psi \in B$, and so

$$\bar{\varphi}(u) \leq \lim \varphi(w_n) \leq \sum_1^\infty \sum_1^\infty \varphi(u_{n,k}) \leq \sum_1^\infty \bar{\varphi}(u_n) + \varepsilon.$$

Thus $\bar{\varphi}$ is B -absolutely continuous.

We are now in a position to prove the main theorem.

Theorem 58.4. *If B is a normal subspace in L^\sim which has locally a countable order basis, then for every $0 < \varphi \in L^\sim$ and every $0 < u \in L$ we have $(P_B\varphi)(u) = \inf (\lim \varphi(u_n): 0 \leq u_n \uparrow \leq u \text{ and } \varphi(u - u_n) \downarrow 0 \text{ for all } 0 < \psi \in B)$.*

Proof. By the preceding theorem we have that $\bar{\varphi}$ is B -absolutely continuous. Since B has locally a countable order basis it follows from Theorem 58.2 that $\bar{\varphi} \in B$, and so $\bar{\varphi} \leq \varphi$ implies that $\bar{\varphi} \leq (P_B\varphi)$. On the other hand, if $0 \leq \varphi' \in B$ and $\varphi' \leq \varphi$, then $\varphi' \leq \bar{\varphi}$, and hence $(P_B\varphi) = \sup (\varphi': 0 \leq \varphi' \leq \varphi \text{ and } \varphi' \in B) \leq \bar{\varphi}$. Thus $P_B\varphi = \bar{\varphi}$ and the proof is finished.

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